On Fox's Conjecture and Quasi-Alternating Knots

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Motivations

Conjecture 1. *(Fox's Strong Trapezoidal Conjecture). Let K be an alternating knot with signature* $|\sigma(K)| = 2k$ *and Alexander polynomial*

$$\Delta_K(t) = \sum_{j=0}^{2n} (-1)^j a_j t^{2n-j}, a_j > 0,$$

then $a_o < a_1 < \cdots < a_{n-m-1} < a_{n-m} = \cdots = a_{n+m} > a_{n+m+1} > \cdots > a_{2n}$*, and moreover* $m \leq k$*.*

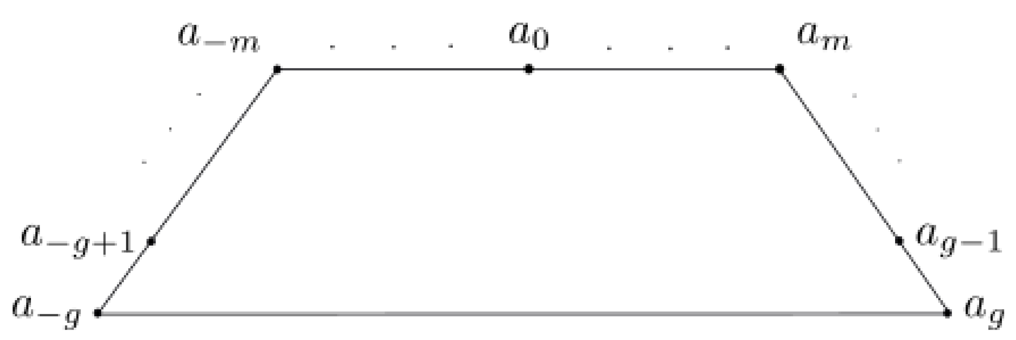

Figure 1: A Visualization of Fox's Conjecture.

The first part of the conjecture (omitting $m \leq k$) is known as the *weak* conjecture. The strong conjecture is true for 2-bridge and genus-2 knots [2]. In this project, we study Fox's Conjecture for certain alternating pretzel knots. We also investigate the conjecture for classes of knots that are not alternating, e.g., almost alternating and quasi-alternating knots.

A Family of Alternating Pretzel Knots

We examine Fox's Conjecture for the family $K_n = P(2, 3, 3+2n),\ n \geq 0$, of alternating pretzel knots formed by adding twists to the third tangle of the $P(2, 3, 3)$ pretzel knot. Jong [5] gives a method to calculate the Alexander Polynomial using the *maximal rooted subtrees* of the *immersed graph* of the knot. This process involves creating a graph from the oriented knot diagram by placing vertices at each crossing, and weighing each edge according to the orientation of the knot at each overcrossing. Weighted subtrees are then created from the immersed graph. This process is shown in Figures 2 and 3 for $K_0 = P(2, 3, 3)$.

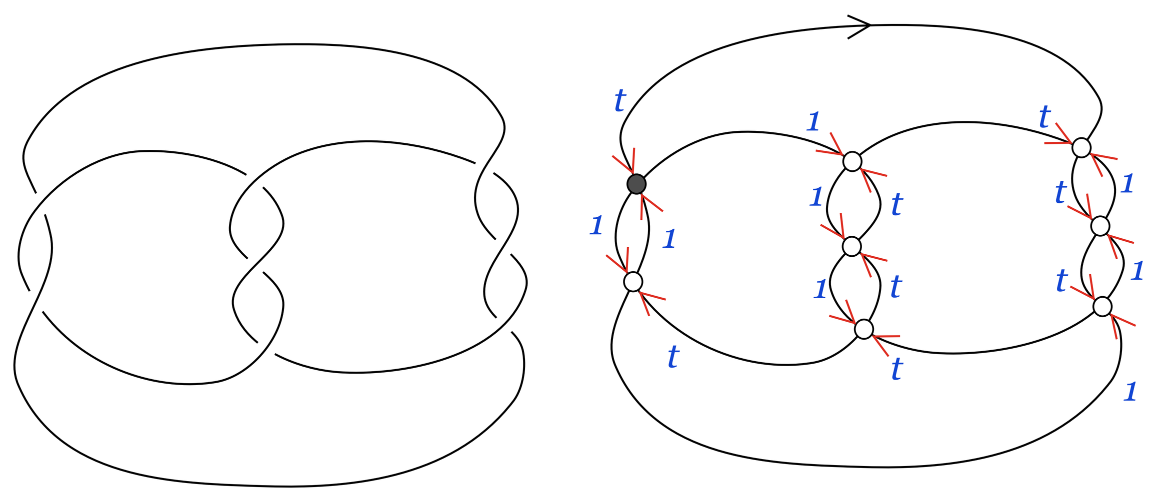

Figure 2: K_0 with its immersed graph (The shaded point is base point c_0 of the subtrees).

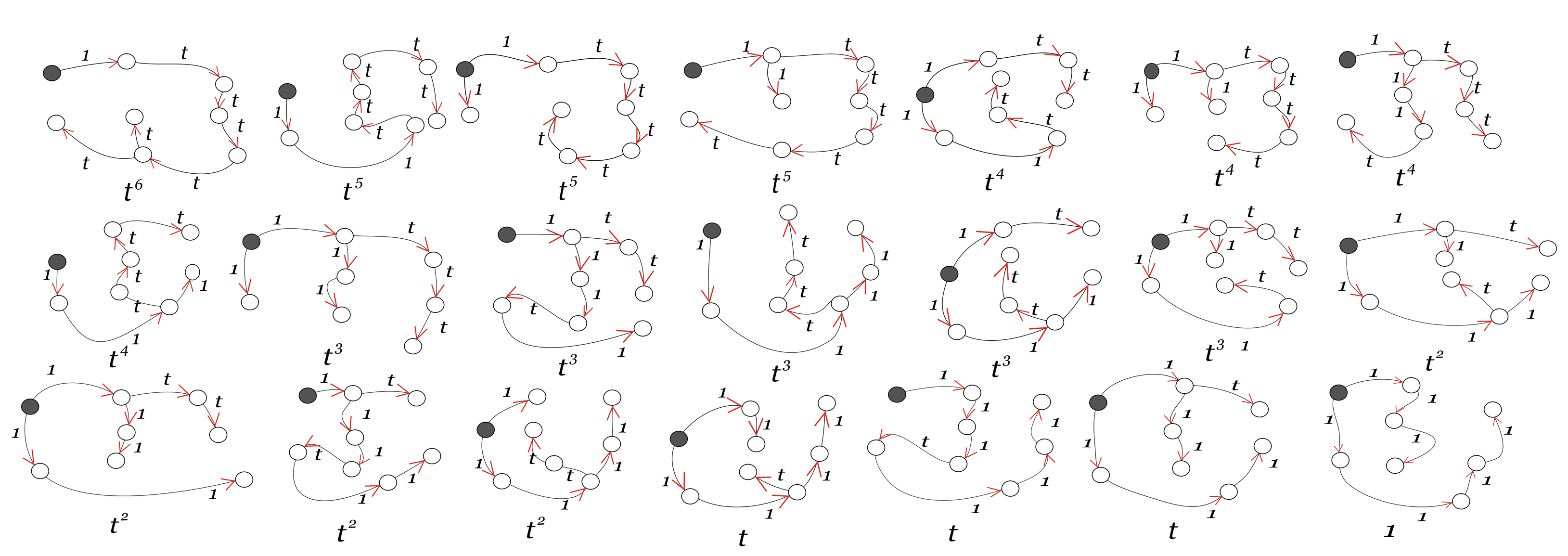

Figure 3: The maximal rooted subtrees (with weights) for K_0.

This forest is denoted $\mathfrak{T}(K_0; c_0)$ The Alexander Polynomial for K_0 can be found by taking the sum of the weights of all the trees, and then evaluating at $-t$.

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Theorem 2. *Fox's Conjecture holds for all* K_n*.*

It can be shown that the cardinality of $\mathfrak{T}(K_n; c_0)$ is $21 + 10n = \det(K_n)$, and all trees in $\mathfrak{T}(K_n; c_0)$ can be built by adding vertices to trees in $\mathfrak{T}(K_0; c_0)$. This allows us to generalize the Alexander Polynomial as follows:

$$\Delta_{K_n}(t) = t^{6+2n} - 3t^{5+2n} + 4t^{4+2n} - 5t^{3+2n} + \cdots \pm 5t^{3+n} \mp 5t^{2+n} \pm \cdots + 4t^2 - 3t + 1.$$

Each coefficient between $5t^{3+2n}$ and $5t^{3+n}$ is equal to ± 5. Similarly, each coefficient after $5t^{2+n}$ and before $4t^2$ is equal to ± 5. One can see from this polynomial that the weak conjecture holds for all K_n, so we now provide a sketch of the proof that Fox's Strong Conjecture holds for all K_n.

To verify the strong conjecture, we show that m is bounded by half of the signature. The value of m corresponds to the number of coefficients $a_i,\ i > n$, of $\Delta_{K_n}(t)$ such that $a_i = a_n$. We have n such coefficients equal to $a_n = 5$, so $n = m$ for all K_n.

Jabuka [4] gives a formula for calculating the signature of pretzel knots. Using this formula we have $|\sigma(K_n)| = 4 + 2n$ for all K_n. We must show that $m \leq k$, where $k = \frac{1}{2} \cdot \sigma(K_n) = 1 + n$. It is easy to see that for all $n \geq 0,\quad m = n \leq 1 + n$, so $m \leq k$, and the strong conjecture holds for all K_n.

Next Frontier: Quasi-Alternating Knots

We next investigate to what extent Fox's conjecture is true for other interesting classes of knots. We will be discussing the set of quasi-alternating knots, which generalizes the notion of alternating knots.

Definition. The set Q of quasi-alternating links is the smallest set of links such that:

- The unknot is in Q.
- If the link L has a diagram D with a crossing c such that:
  1. both smoothings of c, L_o *and* L_∞ are in Q
  2. $\det(L) = \det(L_o) + \det(L_\infty)$

then L is in Q. We say that a crossing c satisfying the properties above is a *quasi-alternating* crossing of the diagram D.

It follows that all alternating knots are quasi-alternating, but the converse is not true. For example, the family of non-alternating pretzel knots $P(2, 3, -2n-1),\ n \geq 1$ is quasi-alternating [1]. It can be shown that Fox's Conjecture holds only for the $n = 1$ case, which is also the only slice knot in the family. Therefore, we ask if the conjecture holds when we restrict to slice knots.

Question. *Does Fox's Conjecture hold for quasi-alternating slice knots?*

We answer this question for a family of quasi-alternating, slice pretzel knots, and for all quasi-alternating slice knots with twelve crossings or less.

Theorem 3. *Fox's Conjecture holds for the family* $P(2n, 2n+1, -2n-1),\ n \geq 1,$*, and for all quasi-alternating slice knots up to twelve crossings (there are forty-three such knots).*

To verify Fox's Conjecture for $P(2n, 2n+1, -2n-1)$, we obtain a generalized Seifert matrix for these knots. See Figure 4.

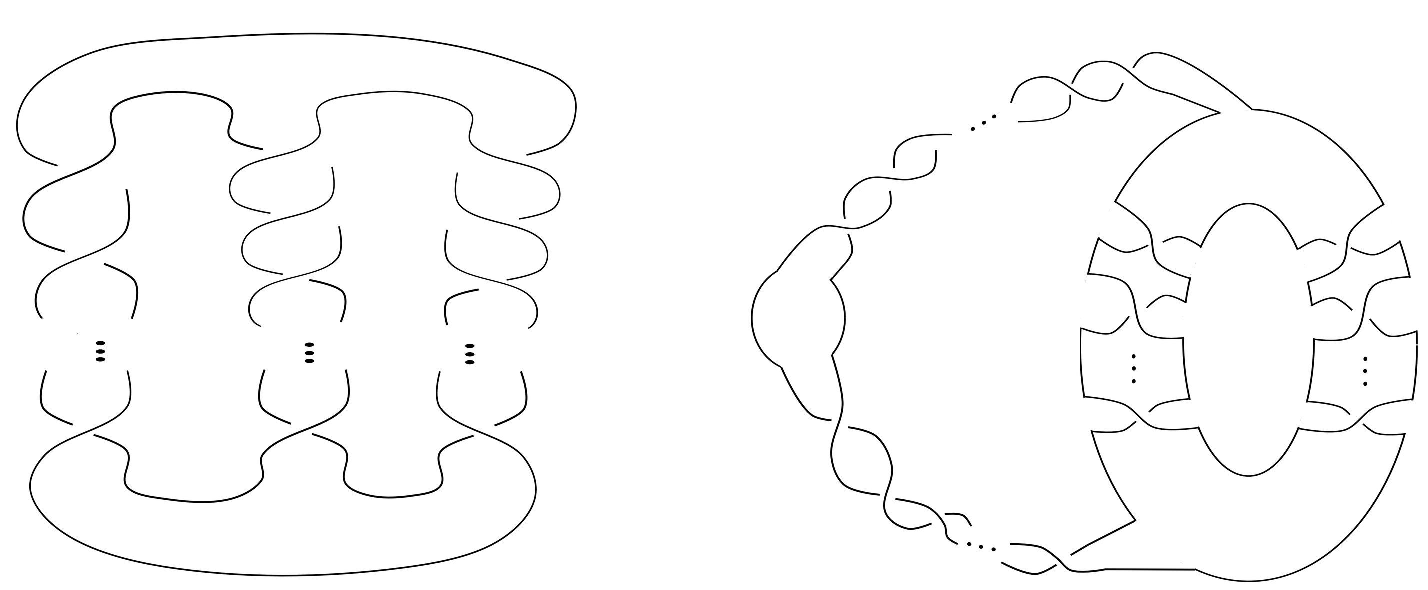

Figure 4: The $P(2n, 2n+1, -2n-1)$ pretzel knot and its Seifert surface.

We notice that if the embedding method stays consistent that corresponding matrices change accordingly. The most noticeable change is in the $a_{n,n}$ entry. This matrix for the n^{th} case is as follows:

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & \cdots & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & \cdots & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -n
\end{pmatrix}$$

From this we calculate a generalized Alexander Polynomial in terms of n as follows:

$$t^{2n} - 2t^{2n-1} + 3t^{2n-2} - \cdots - (2n)t + (2n+1) - (2n)t^{-1} + \cdots + 3t^{-2n+2} - 2t^{-2n+1} + t^{-2n}$$

Future Work

There are many questions that naturally follow from this such as:

1. ***What information do sign alternating Alexander Polynomials tell us about their respective knots?***
   Throughout our research we have discovered that the alternating sign pattern appears in almost alternating knots as well. We have reason to believe this alternation may help us rigorously define alternating knots.
2. ***Do all quasi-alternating slice knots satisfy Fox's Conjecture?***
   Our findings suggest this is true, as it holds for the family $P(2n, 2n+1, -2n-1)$, and for slice knots of twelve crossings or less, by calculation using Jablan's tables of quasi-alternating knots. [3]
3. ***A Geometric Approach to Fox's Conjecture***
   Our work on Fox's Conjecture has been largely computation based. J. Greene provides a geometric explanation for an alternating link; perhaps this may be used to further our understanding of Fox's Conjecture.

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